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J. Phys. A: Math. Gen. 28 (1995) 7233–7242. Printed in the UK

Bogoliubov transformation technique and Gaussian wavefunctional approach for a class of QFTS

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Received 22 March 1995, in final form 17 July 1995

Abstract. It is shown that for a $(D+1)$-dimensional scalar field system with an arbitrary potential whose Fourier representation exists in the sense of tempered distributions, the effective potential and multi-particle-state energies by the Bogoliubov transformation technique are identical to those calculated by the Gaussian wavefunctional approach. However, the Bogoliubov transformation technique differs from the latter as it can be applied to quantizing a static soliton without difficulty.

1. Introduction

Bogoliubov transformation [1] is a unitary transformation from one vacuum to another. With the aid of this technique, the transformed vacuum, a non-perturbative vacuum, has been defined to investigate vacuum structure or phase transition of many models in $(1+1)$ dimensions, such as the Thirring model [2,3], the Gross–Neveu model [4,5], QED+NJL model [6], massive Schwinger model [7], the O(N) nonlinear σ model [8], and the sine–Gordon model [9,10]. In the meantime, the Gaussian wavefunctional approach (GWFA) [11,12] has also been used to discuss vacuum structures, bound states, scattering phase shifts, solitons and phase transitions in various models—the scalar, Fermi and gauge fields or their interactions [13–21]. Recently, the equivalence between these two methods has been shown by calculating the effective potential or one- and two-particle excited-state energies in the $(1+1)$-dimensional $\lambda\phi^4$ model [2] and in the $(D+1)$-dimensional sine–Gordon and sinh–Gordon models [22].

In this paper, we intend to compare the above two methods by considering a relatively general model with the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi_x\partial^\mu\phi_x - V(\phi_x) \tag{1}$$

where $\phi_x \equiv \phi(x)$, and the potential $V(\phi_x)$ has a Fourier representation in a sense of tempered distributions [23]. The potentials of many models, for example the various polynomial models and the sine–Gordon and sinh–Gordon models, have this property. We shall calculate the effective potential, one- as well as two-particle energies for equation (1), and generalize the Bogoliubov transformation technique (BTT) to the non-uniform background case. We find that the results of the above energies from the BTT are correspondingly identical to those from the GWFA. One can also see that in the framework of each of these two methods the different models have almost the same expressions for some physical quantities. However, although there is a weak point when a non-uniform background field is introduced into the GWFA [20], the drawback disappears when one employs the BTT.

We shall first use the BTT in section 2, and then the GWFA in section 3 to evaluate the effective potential one- and two-particle energies for system (1). In section 4, we discuss the non-uniform background case. Finally, we shall discuss our results.

2. Bogoliubov transformation technique

In the functional Schrödinger picture, the Hamiltonian operator corresponding to equation (1) reads

$$H = \int_x \mathcal{H}_x = \int_x \left\{ \frac{1}{2}\Pi_x^2 + \frac{1}{2}(\nabla\phi_x)^2 + V(\phi_x) \right\} \tag{2}$$

with ϕ_x the quantum counterpart of the field, $\Pi_x \equiv -\mathrm{i}\frac{\delta}{\delta\phi_x}$ conjugate to ϕ_x, $\int_x \equiv \int \mathrm{d}^D x = \int \mathrm{d}x_1\,\mathrm{d}x_2 \ldots \mathrm{d}x_D$, and ∇ the gradient operator in D-dimensional $\boldsymbol{x}$-space. The fundamental operators ϕ_x and Π_x can be expanded in terms of creation and annihilation operators based on the naive vacuum $|0\rangle$ with respect to the bare mass parameter m in equation (2) without an interaction as

$$\phi_x = \int \frac{\mathrm{d}^D p}{\sqrt{(2\pi)^D}} \left[\frac{1}{2\omega(p,m)} \right]^{1/2} [a(\boldsymbol{p},m)\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} + a^\dagger(\boldsymbol{p},m)\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}] \tag{3}$$

and

$$\Pi_x = -\mathrm{i}\int \frac{\mathrm{d}^D p}{\sqrt{(2\pi)^D}} \left[\frac{\omega(p,m)}{2} \right]^{1/2} [a(\boldsymbol{p},m)\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} - a^\dagger(\boldsymbol{p},m)\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}] \tag{4}$$

where $\boldsymbol{p}$ represents a D-dimensional vector, $p = |\boldsymbol{p}|$, $\omega(p,m) = \sqrt{p^2+m^2}$, $a(\boldsymbol{p},m)|0\rangle = 0$ and $[a(\boldsymbol{p},m), a^\dagger(\boldsymbol{p}',m)] = \delta(\boldsymbol{p}-\boldsymbol{p}')$.

Employing the Bogoliubov transformation, one can define the new operators

$$\begin{aligned}\begin{bmatrix} b(\boldsymbol{p}) \\ b^\dagger(-\boldsymbol{p}) \end{bmatrix} &= \begin{bmatrix} Ua(\boldsymbol{p},m)U^{-1} \\ Ua^\dagger(-\boldsymbol{p},m)U^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \cosh[\theta(\boldsymbol{p})] & -\sinh[\theta(\boldsymbol{p})] \\ -\sinh[\theta(\boldsymbol{p})] & \cosh[\theta(\boldsymbol{p})] \end{bmatrix} \begin{bmatrix} a(\boldsymbol{p},m) - \sqrt{\frac{1}{2}\omega(p,m)}g(\boldsymbol{p}) \\ a^\dagger(-\boldsymbol{p},m) - \sqrt{\frac{1}{2}\omega(p,m)}g(\boldsymbol{p}) \end{bmatrix}\end{aligned} \tag{5}$$

with $U = U_1U_2$, where $U_j = \exp(A_j^\dagger - A_j)$, $A_j^\dagger (j = 1, 2)$ being

$$A_1^\dagger = \int \mathrm{d}^D p \sqrt{\frac{\omega(p,m)}{2}} g(\boldsymbol{p}) a^\dagger(\boldsymbol{p},m) \tag{6}$$

and

$$A_2^\dagger = \frac{1}{2}\int \mathrm{d}^D p\, \theta(\boldsymbol{p}) a^\dagger(\boldsymbol{p},m) a^\dagger(-\boldsymbol{p},m). \tag{7}$$

Here the parameters $g(\boldsymbol{p})$ and $\theta(\boldsymbol{p})$ are real parameters to be determined. One can check the equal-time commutation relation

$$[b(\boldsymbol{p}), b^\dagger(\boldsymbol{p}')] = \delta(\boldsymbol{p}-\boldsymbol{p}'). \tag{8}$$

Obviously, the inverse of equation (5) is

$$\begin{bmatrix} a(\boldsymbol{p},m) \\ a^\dagger(-\boldsymbol{p},m) \end{bmatrix} = \begin{bmatrix} \cosh[\theta(\boldsymbol{p})] & \sinh[\theta(\boldsymbol{p})] \\ \sinh[\theta(\boldsymbol{p})] & \cosh[\theta(\boldsymbol{p})] \end{bmatrix} \begin{bmatrix} b(\boldsymbol{p}) \\ b^\dagger(-\boldsymbol{p}) \end{bmatrix} + \begin{bmatrix} \sqrt{\frac{1}{2}\omega(p,m)}g(\boldsymbol{p}) \\ \sqrt{\frac{1}{2}\omega(p,m)}g(\boldsymbol{p}) \end{bmatrix}. \tag{9}$$

The unitary operator U can transform the bare vacuum $|0\rangle$ into a non-perturbative trial vacuum [2], the Bogoliubov vacuum,

$$|\tilde{0}\rangle = U|0\rangle = U_1 U_2|0\rangle \tag{10}$$

where $g(p)$ and $\theta(p)$ will be regarded as variational parameters and determined by *variationally extremizing the expectation value of* H *in* $|\tilde{0}\rangle$. *The translational invariance* requires that $g(p)$ and $\theta(p)$ are even functions of p. Although $a(p,m)|\tilde{0}\rangle \neq 0$, it is evident that $b(p)|\tilde{0}\rangle = 0$. Therefore, from equation (8), $b^\dagger(p)$ is the creation operator of a quasi-particle with the momentum p.

For $|\tilde{0}\rangle$, one can easily evaluate

$$\langle\tilde{0}|\hat{P}|\tilde{0}\rangle = 0 \tag{11}$$

$$\langle\tilde{0}|\phi_x|\tilde{0}\rangle = \int \frac{\mathrm{d}^D p}{\sqrt{(2\pi)^D}} g(p)\mathrm{e}^{-\mathrm{i}p\cdot x} \tag{12}$$

where $\hat{P} = -\int_x \Pi_x \nabla\phi_x$ is the total momentum operator of the field system (1). Equation (12) indicates that $g(p)$ is the Fourier component of the vacuum expectation value of ϕ_x. Thus U_1 plays the role of shifting the field system. In order to obtain the effective potential, we take $g(p) = \varphi\delta(p)$, and so

$$\langle\tilde{0}|\phi_x|\tilde{0}\rangle = \varphi \tag{13}$$

with φ the uniform background field.

Acting on the vacuum $|\tilde{0}\rangle$ by $b^\dagger(p)$, one can manufacture multi-particle excited states [11, 24, 12]. For instance, one- and two-particle states can be defined separately as

$$|1\rangle = b^\dagger(p)|\tilde{0}\rangle_{\mathrm{r}} \tag{14}$$

and

$$|2\rangle = \int \mathrm{d}^D p\,\Sigma(p) b^\dagger(p) b^\dagger(-p)|\tilde{0}\rangle_{\mathrm{r}} \tag{15}$$

where $\Sigma(p)$ is the wavefunction in the momentum space and the subscript r means the vacuum is a renormalized one.

According to Coleman's normal ordering prescription [25, 26], we normal-order $\mathcal{H}_x$, *with respect to the normal-ordering mass* Q, *as*

$$\mathcal{N}_Q[\mathcal{H}_x] = \tfrac{1}{2}\Pi_x^2 + \tfrac{1}{2}(\nabla\phi_x)^2 - \tfrac{1}{4}J[Q^2] + \mathcal{N}_Q[V(\phi_x)]$$

with

$$J[Q^2] = \int \frac{\mathrm{d}^D p}{(2\pi)^D}\frac{2p^2 + Q^2}{\sqrt{p^2 + Q^2}}.$$

Using equations (3), (4), (8), (9), and (13), one has

$$\begin{aligned}\langle\tilde{0}|\tfrac{1}{2}[\Pi_x^2 + (\nabla\phi_x)^2]|\tilde{0}\rangle &= \frac{1}{4}\int \frac{\mathrm{d}^D p}{(2\pi)^D}\omega(p,m)(\cosh[\theta(p)] - \sinh[\theta(p)])^2 \\ &\quad + \frac{1}{4}\int \frac{\mathrm{d}^D p}{(2\pi)^D}\frac{p^2}{\omega(p,m)}(\cosh[\theta(p)] + \sinh[\theta(p)])^2.\end{aligned} \tag{16}$$

In order to calculate $\langle\tilde{0}|\mathcal{N}_Q[V(\phi_x)]|\tilde{0}\rangle$, we make the Fourier transformation (at least in the sense of tempered distributions)

$$V(\phi_x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\Omega}{\sqrt{2\pi}}\tilde{V}(\Omega)\mathrm{e}^{\mathrm{i}\Omega\phi_x}. \tag{17}$$

Noticing the Baker–Haussdorf formula

$$e^{A+B} = e^{A}e^{B}e^{-\frac{1}{2}[A,B]} \tag{18}$$

with $[A, B]$ some c-number, and the integral $\int_0^\infty \frac{2a}{\sqrt{\pi}} e^{-a^2x^2}\,dx = 1$, we arrive at

$$\begin{aligned}\langle\tilde{0}|\mathcal{N}_Q[V(\phi_x)]|\tilde{0}\rangle &= \int \frac{d\Omega}{\sqrt{2\pi}} \tilde{V}(\Omega) e^{i\Omega\varphi} \exp\left\{-\frac{1}{4}\Omega^2(I[\theta] - I[Q^2])\right\} \\ &= \int_{-\infty}^{\infty} d\alpha \frac{1}{2\sqrt{\pi}} e^{-\alpha^2/4} V\left(\frac{\alpha}{2}\sqrt{I[\theta] - I[Q^2]} + \varphi\right)\end{aligned} \tag{19}$$

where

$$I[\theta] = \int \frac{d^D p}{(2\pi)^D \omega(p, m)} (\cosh[\theta(p)] + \sinh[\theta(p)])^2 \tag{20}$$

and

$$I[Q^2] = \int \frac{d^D p}{(2\pi)^D} \frac{1}{\sqrt{p^2 + Q^2}}.$$

Collecting equations (16) and (19), one has the energy density of $|\tilde{0}\rangle$

$$\begin{aligned}\mathcal{E}[\varphi, \theta] = \langle\tilde{0}|\mathcal{N}_Q[\mathcal{H}_x]|\tilde{0}\rangle &= \frac{1}{4}\int \frac{d^D p}{(2\pi)^D} \omega(p, m)(\cosh[\theta(p)] - \sinh[\theta(p)])^2 \\ &+ \frac{1}{4}\int \frac{d^D p}{(2\pi)^D} \frac{p^2}{\omega(p, m)} (\cosh[\theta(p)] + \sinh[\theta(p)])^2 - \frac{1}{4} J[Q^2] \\ &+ \int_{-\infty}^{\infty} \frac{d\alpha}{2\sqrt{\pi}} e^{-\alpha^2/4} V\left(\frac{\alpha}{2}\sqrt{I[\theta] - I[Q^2]} + \varphi\right).\end{aligned} \tag{21}$$

Minimizing the energy density with respect to the function $\theta(p)$, we obtain

$$\theta(p, \varphi) = \ln \frac{p^2 + m^2}{p^2 + M^2(\varphi)} \tag{22}$$

where

$$M^2(\varphi) = \int_{-\infty}^{\infty} \frac{d\alpha}{2\sqrt{\pi}} e^{-\alpha^2/4} V^{(2)}\left(\frac{\alpha}{2}\sqrt{I[M^2(\varphi)] - I[Q^2]} + \varphi\right) \tag{23}$$

with

$$V^{(n)}(z) = \frac{d^n V(z)}{dz^n} = \int_{-\infty}^{\infty} \frac{d\Omega}{\sqrt{2\pi}} (i\Omega)^n \tilde{V}(\Omega) e^{i\Omega z} \tag{24}$$

the dependence upon φ being imposed on the Bogoliubov angle $\theta(p)$. Hence, equation (20) becomes

$$I[\theta] = I[M^2(\varphi)] = \int \frac{d^D p}{(2\pi)^D} \frac{1}{\sqrt{p^2 + M^2(\varphi)}}. \tag{25}$$

Consequently, the effective potential, defined as the minimized energy density, is

$$\mathcal{V}(\varphi) = \frac{1}{4}(J[M^2(\varphi)] - J[Q^2]) + \int_{-\infty}^{\infty} \frac{d\alpha}{2\sqrt{\pi}} e^{-\alpha^2/4} V\left(\frac{\alpha}{2}\sqrt{I[M^2(\varphi)] - I[Q^2]} + \varphi\right). \tag{26}$$

Furthermore, the same calculations can be done for the states $|1\rangle$ and $|2\rangle$, and so one finds the one-particle energy

$$m_1 \approx \frac{\langle 1|H|1\rangle}{\langle 1|1\rangle} - \int_x \mathcal{V}(\varphi_0) = \sqrt{p^2 + M^2(\varphi_0)} \tag{27}$$

as well as the two-particle energy

$$m_2 = \frac{\langle 2|H|2\rangle}{\langle 2|2\rangle} - \int_x \mathcal{V}(\varphi_0) = \left\{2\int \mathrm{d}^D p[\Sigma(p)]^2\sqrt{p^2+M^2} + \frac{\nu^{(4)}(\varphi_0)}{8(2\pi)^D}\left[\int \frac{\Sigma(p)\,\mathrm{d}^D p}{\sqrt{p^2+M^2(\varphi_0)}}\right]^2\right\} \Big/ \int \mathrm{d}^D p[\Sigma(p)]^2 \tag{28}$$

where

$$\nu^{(n)}(z) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha}{2\sqrt{\pi}} \mathrm{e}^{-\alpha^2/4} V^{(n)}\left(\frac{\alpha}{2}\sqrt{I[M^2(z)] - I[Q^2]} + z\right) \tag{29}$$

and φ_0 corresponds to $|\tilde{0}\rangle_r$. From equation (27), one can see that $M^2(\varphi_0)$ is just the mass of a quasi-particle. Moreover, the two terms in equation (28) can be interpreted as the kinetic energy of the two constituent particles and their interacting energy, respectively. It is evident that the signs of $\nu^{(4)}(\varphi_0)$ determine whether the interacting force between two particles is attractive or repulsive, and when $\nu^{(4)}(\varphi_0) = 0$ in equation (28) there are no interacting effects between the two particles. One perhaps has also noticed that the further analysis of equation (28) can give the two-particle bound-state mass and the scattering phase shifts [18, 19].

3. Gaussian wavefunctional approach

For the GWFA, the vacuum ansatz reads [14]

$$|\varphi\rangle = N_f \exp\left\{\mathrm{i}\int \mathcal{P}_x\phi_x - \frac{1}{2}\int_{x,y}(\phi_x - \varphi_x) f_{xy}(\phi_y - \varphi_y)\right\} \tag{30}$$

where $\mathcal{P}_x$, φ_x and f_{xy} are the variational parameters. N_f is some normalization constant, and depends upon f_{xy}. Due to the translational invariance of the vacuum, it is necessary for $f_{xy} = f_{yx}$. Besides, the inverse f_{xy}^{-1} of f_{xy} has to exist, i.e.

$$\int_z f_{xy} f_{yz}^{-1} = \delta_{xz} \tag{31}$$

must be true. It is easy to show

$$\langle\varphi|\phi_x|\varphi\rangle = \varphi_x \tag{32}$$

and

$$\langle\varphi|\Pi_x|\varphi\rangle = \langle\varphi| - \mathrm{i}\frac{\delta}{\delta\phi_x}|\varphi\rangle = \mathcal{P}_x. \tag{33}$$

Making the Fourier transformation

$$f_{yz} = \int \frac{\mathrm{d}^D p}{(2\pi)^D} f(p)\mathrm{e}^{\mathrm{i}p\cdot(y-z)} \tag{34}$$

and using functional integration technique, one can easily evaluate

$$\langle\varphi|\frac{1}{2}[\Pi_x^2 + (\nabla\phi_x)^2]|\varphi\rangle = \frac{1}{2}\mathcal{P}_x^2 + \frac{1}{2}(\nabla\varphi_x)^2 + \frac{1}{4}\int \frac{\mathrm{d}^D p}{(2\pi)^D} f(p) + \frac{1}{4}\int \frac{\mathrm{d}^D p}{(2\pi)^D}\frac{p^2}{f(p)} \tag{35}$$

and

$$\langle\varphi|\mathcal{N}_Q[V(\phi_x)]|\varphi\rangle = \int_{-\infty}^{\infty} \mathrm{d}\alpha \frac{1}{2\sqrt{\pi}} \mathrm{e}^{-\alpha^2/4} V\left(\frac{\alpha}{2}\sqrt{f_{xx}^{-1} - I[Q^2]} + \varphi_x\right). \tag{36}$$

Thus, the energy density of $|\varphi\rangle$ is

$$\mathcal{E}[\varphi,\mathcal{P},f]=\langle\varphi|\mathcal{N}_Q[\mathcal{H}_x]|\varphi\rangle=\frac{1}{2}\mathcal{P}_x^2+\frac{1}{2}(\nabla\varphi_x)^2-\frac{1}{4}J[Q^2]+\frac{1}{4}\int\frac{\mathrm{d}^Dp}{(2\pi)^D}f(p)$$
$$+\frac{1}{4}\int\frac{\mathrm{d}^Dp}{(2\pi)^D}\frac{p^2}{f(p)}+\int_{-\infty}^{\infty}\mathrm{d}\alpha\frac{1}{2\sqrt{\pi}}\mathrm{e}^{-\alpha^2/4}V\left(\frac{\alpha}{2}\sqrt{f_{xx}^{-1}-J[Q^2]}+\varphi_x\right). \quad (37)$$

Taking $\varphi_x=\text{constant}=\varphi$ and minimizing the energy density with respect to the function $\mathcal{P}_x$ and $f(p)$, respectively, we obtain $\mathcal{P}_x=0$ and

$$f(p)=\sqrt{p^2+\mu^2(\varphi)} \quad (38)$$

with

$$\mu^2(\varphi)=\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha}{2\sqrt{\pi}}\mathrm{e}^{-\alpha^2/4}V^{(2)}\left(\frac{\alpha}{2}\sqrt{f_{xx}^{-1}-I[Q^2]}+\varphi\right). \quad (39)$$

Here,

$$f_{xx}^{-1}=\int\frac{\mathrm{d}^Dp}{(2\pi)^D}\frac{1}{\sqrt{p^2+\mu^2(\varphi)}}=I[\mu^2(\varphi)].$$

Hence, the effective potential is

$$\mathcal{V}(\varphi)=\frac{1}{4}(J[\mu^2(\varphi)]-J[Q^2])+\int_{-\infty}^{\infty}\mathrm{d}\alpha\frac{1}{2\sqrt{\pi}}\mathrm{e}^{-\alpha^2/4}V\left(\frac{\alpha}{2}\sqrt{I[\mu^2(\varphi)]-I[Q^2]}+\varphi\right). \quad (40)$$

In order to construct the excited states, following Barnes and Ghandour [24], one can manufacture the annihilation and creation operators with respect to the renormalized vacuum $|\varphi_0\rangle$ of the trial vacuum $|\varphi\rangle$

$$A_f(p)=\left(\frac{1}{2(2\pi)^Df(p)}\right)^{1/2}\int_x\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}\left[f(p)(\phi_x-\varphi_0)+\frac{\delta}{\delta\phi_x}\right] \quad (41)$$

and

$$A_f^\dagger(p)=\left(\frac{1}{2(2\pi)^Df(p)}\right)^{1/2}\int_x\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}\left[f(p)(\phi_x-\varphi_0)-\frac{\delta}{\delta\phi_x}\right]. \quad (42)$$

Then, multi-particle excited states can be constructed by $A_f^\dagger(p)$ acting on $|\varphi_0\rangle$, such as the one-particle state

$$|1\rangle=A_f^\dagger(p)|\varphi_0\rangle \quad (43)$$

and the S-wave two-particle state

$$|2\rangle=\int\mathrm{d}^Dp\Sigma(p)A_f^\dagger(p)A_f^\dagger(-p)|\varphi\rangle \quad (44)$$

where $\Sigma(p)$ is the Fourier transformation of the S-wave function of the two particles. Consequently, one finds the energy of the one-particle

$$m_1=f(p)=\sqrt{p^2+\mu^2(\varphi_0)} \quad (45)$$

and the energy of the two-particle system

$$m_2=\left\{2\int\mathrm{d}^Dp(\Sigma(p))^2\sqrt{p^2+\mu^2(\varphi_0)}+\frac{v^{(4)}(\varphi_0)}{8(2\pi)^D}\left(\int\mathrm{d}^Dp\frac{\Sigma(p)}{f(p)}\right)^2\right\}\Big/\int\mathrm{d}^Dp(\Sigma(p))^2. \quad (46)$$

Now we have briefly given the effective potential, one- and two-particle energies for model (1) by the GWFA. Obviously, these expressions are correspondingly identical to those obtained by the BTT in the previous section. Note that for $D<3$ these expressions have no divergences, because the integrals $J[M^2(\varphi)]-J[Q^2]$, $I[M^2(\varphi)]-I[Q^2]$, $J[\mu^2(\varphi)]-J[Q^2]$, and $I[\mu^2(\varphi)]-I[Q^2]$ are all finite for $D<3$. Consequently, the BTT gives the same physical results as the GWFA.

By way of explanation and justification, we now consider the general ϕ^6 model with the potential

$$V_1(\phi_x)=a_{\rm B}\phi_x+\tfrac{1}{2}m_{\rm B}^2\phi_x^2+g_{\rm B}\phi_x^3+\lambda_{\rm B}\phi_x^4+\xi\phi_x^6 \tag{47}$$

and the sinh–Gordon model with the potential

$$V_2(\phi_x)=\frac{m^2}{\gamma^2}[\cosh(\gamma\phi_x)-1] \tag{48}$$

which is that of the sine–Gordon model when $\gamma^2=-\beta^2$, using formulae (23), (26), (27) and (28). In the sense of tempered distributions, $V_1(\phi_x)$ and $V_2(\phi_x)$ have their own Fourier representations [23]. For the ϕ^6 model, noting that

$$\int_{-\infty}^{\infty}\alpha^{2n+1}\mathrm{e}^{-\alpha^2/4}\frac{\mathrm{d}\alpha}{2\sqrt{\pi}}=0 \qquad \text{and} \qquad \int_{-\infty}^{\infty}\alpha^{2n}\mathrm{e}^{-\alpha^2/4}\frac{\mathrm{d}\alpha}{2\sqrt{\pi}}=2^n\cdot 1\cdot 3\cdot 5\cdots(2n-1)$$

one can have ($I\equiv I[M^2(\varphi)]-I[Q^2]$, $J\equiv J[M^2(\varphi)]-J[Q^2]$)

$$M^2(\varphi)=m_{\rm B}^2+6g_{\rm B}\varphi+12\lambda_{\rm B}(\tfrac{1}{2}I+\varphi^2)+30\xi(\varphi^4+3I\varphi^2+\tfrac{3}{4}I^2) \tag{49}$$

$$v^{(4)}(\varphi_0)=2\cdot 12[\lambda_{\rm B}+15\xi(\tfrac{1}{2}I+\varphi_0^2)] \tag{50}$$

and

$$\begin{aligned}\mathcal{V}(\varphi)=\tfrac{1}{4}J+a_{\rm B}\varphi+\tfrac{1}{2}m_{\rm B}^2(\tfrac{1}{2}I+\varphi^2)+g_{\rm B}(\tfrac{3}{2}I\varphi+\varphi^3)+\lambda_{\rm B}(\tfrac{3}{4}I^2+3I\varphi^2+\varphi^4)\\ +\xi(\tfrac{15}{8}I^3+\tfrac{45}{4}I^2\varphi^2+\tfrac{15}{2}I\varphi^4+\varphi^6).\end{aligned} \tag{51}$$

Substituting equation (49) into equation (27) and equations (49), (50) into equation (28) can give m_1 and m_2, respectively. When $\xi=0$ and $D=3$, equation (51) is the same as in equation (5) in [27] if I is replaced by $I[M^2(\varphi)]$ and J by $J[M^2(\varphi)]$. In [27], equation (5) is unrenormalized. In the above, if I is replaced by $I[M^2(\varphi)]$ and J by $J[M^2(\varphi)]$, the results are unrenormalized ones, too. Thus our formulae can give the same results as in [27]. Furthermore, the above $\mathcal{V}(\varphi)$, m_1 and m_2 are also reduced to those in [28] for $a_{\rm B}=g_{\rm B}=0$ and in [2] and [13] for $a_{\rm B}=g_{\rm B}=\xi=0$. As for the sinh–Gordon model, we have

$$M^2(\varphi)=\int_{-\infty}^{\infty}m^2\cosh\left[\gamma\left(\frac{\alpha}{2}\sqrt{I}+\varphi\right)\right]\mathrm{e}^{-\alpha^2/4}\frac{\mathrm{d}\alpha}{2\sqrt{\pi}}=m^2\exp\left\{\frac{\gamma^2}{4}I\right\}\cosh(\gamma\varphi) \tag{52}$$

$$v^{(4)}(\varphi_0)=m^2\gamma^2\exp\left\{\frac{\gamma^2}{4}I\right\}\cosh(\gamma\varphi_0) \tag{53}$$

and

$$\mathcal{V}(\varphi)=\frac{1}{4}J+\frac{m^2}{\gamma^2}\left[\exp\left\{\frac{\gamma^2}{4}I\right\}\cosh(\gamma\varphi)-1\right]. \tag{54}$$

From equations (27), (28), (52) and (53), one can have m_1 and m_2 for the sinh–Gordon model. These results (with some extra algebraic treatments) are identical to those in [14, 19, 22].

Throughout section 2 and this section, we have generally shown the equivalence between the BTT and the GWFA. This can be understandable, for the ansatz (10) is indeed the Gaussian-type state analogous to equation (30). Therefore it is not difficult to see that this agreement also occurs at the other multi-particle energies and may remain valid for the theories concerning the Fermi and gauge fields. In this paper we shall not continue to discuss them. Instead, in the following section, we shall extend the BTT to the non-uniform background. By the way, section 2 and this section have already shown that in the framework of the BTT (or GWFA), different models will have almost the same expression for m_1 or m_2.

4. Non-uniform background

In this section we introduce a non-trivial background field into the BTT for it to be used in quantum solitons or instantons, and we are only interested in the ground-state case.

In fact, this extension is straightforward. One can set

$$\langle\tilde{0}|\phi_x|\tilde{0}\rangle = \varphi_x \tag{55}$$

instead of equation (13), i.e. $g(p)$ is regarded as the Fourier component of φ_x. This only results in replacing the constant φ by the space-dependent function φ_x in section 2, except for adding the extra term $\frac{1}{2}(\nabla\varphi_x)^2$ to equations (16), (21) and (26). Nevertheless, the variationally-extremized procedure with respect to θ makes θ in equation (5) space-dependent, i.e. $\theta(p) \to \theta(p, x)$. Owing to the space dependence of θ, $b(p)$ and $b^\dagger(p)$ became space dependent, and so one has the relation

$$[b(p,x), b^\dagger(p',x')] = (\cosh^2[\theta(p,x)] - \sinh^2[\theta(p',x')])\delta(p-p'). \tag{56}$$

This appears to bring about an inconsistency in the calculations implemented before the variationally-extremized procedure, i.e. it appears that by substituting $g(p)$ in equation (55) and $\theta(p,x)$ into equation (5) and re-calculating the energy density, one should obtain a much more complicated expression than equation (21), instead of the above-mentioned result which is analogous to equation (21). Fortunately, in the course of calculating the energy density, one encounters only the commutator $[b(p,x), b^\dagger(p',x)]$, which is $\delta(p-p')$. Hence, the energy can be read as

$$\begin{aligned}E_0(\varphi) = \int_x \langle\tilde{0}|\mathcal{N}_Q[\mathcal{H}_x]|\tilde{0}\rangle = \int_x \bigg\{ &\frac{1}{2}(\nabla\varphi_x)^2 + \frac{1}{2}\int \frac{\mathrm{d}^D p}{(2\pi)^D}\sqrt{p^2 + M^2(\varphi_x)} \\ &-\frac{1}{4}M^2(\varphi_x)\int \frac{\mathrm{d}^D p}{(2\pi)^D}\frac{1}{\sqrt{p^2+M^2(\varphi_x)}} - \frac{1}{4}J[Q^2] \\ &+\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha}{2\sqrt{\pi}}\mathrm{e}^{-\alpha^2/4}V\left(\frac{\alpha}{2}\sqrt{I[M^2(\varphi_x)] - I[Q^2]} + \varphi_x\right)\bigg\}\end{aligned} \tag{57}$$

the integrand of which still resembles equation (26). Thus, the spatial dependence of the background field would not hinder our generalization, though θ is only regarded as a function of momentum p in the original ansatz (10).

Minimizing E_0 with respect to φ_x or $g(p)$, we obtain the static equation

$$\nabla^2\varphi_x - \int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha}{2\sqrt{\pi}}\mathrm{e}^{-\alpha^2/4}V^{(1)}\left(\frac{\alpha}{2}\sqrt{I[M^2(\varphi_x)] - I[Q^2]} + \varphi_x\right) = 0. \tag{58}$$

This equation is just the static version of equation (2.37) in [29]. It can be used for quantizing a static soliton or an instanton in $(D-1)$ space dimensions.

As an example, we consider the sine–Gordon model (the potential is $V_2(\phi_x)$ with $\gamma^2 = -\beta^2$)

$$\mathcal{L}_x = \frac{1}{2}\partial_\mu\phi_x\partial^\mu\phi_x + \frac{m^2}{\beta^2}[\cos(\beta\phi_x) - 1] \tag{59}$$

where m and β are the bare parameters. According to equation (58), one has

$$\nabla^2\varphi_x - \frac{m^2}{\beta}\exp\left\{-\frac{\beta^2}{4}(I[M^2(\varphi_x)] - I[Q^2])\right\}\sin(\beta\varphi_x) = 0. \tag{60}$$

In (1 + 1) dimensions, we get

$$\frac{\mathrm{d}^2\varphi_x}{\mathrm{d}x^2} - \frac{m_r^2}{\beta}(\cos(\beta\varphi_x))^{8\pi/(8\pi-\beta^2)}\sin(\beta\varphi_x) = 0 \tag{61}$$

with $m_r^2 = M^2(\varphi_x = 0)$. This is simply equation (6) in [30] (static case).

In the last two sections, we have seen the equivalence between the BTT and the GWFA in a constant background. However, in the space-dependent background, the BTT seems better than the GWFA. As was pointed out in [20], as in the last section, the variationally-minimized step in the GWFA imposes a φ_x-dependence on the Fourier component $f(p)$ of the quantum fluctuation f_{xy} and so $f(p)$ will become space-dependent. Consequently, equation (31) no longer holds rigorously [20]. Nevertheless, whether equation (31) holds or not is just a question of whether the generalization of the GWFA to a non-uniform background is self-consistent. Regardless of this problem, after the minimized step, if one continues to advance with the GWFA, then no further difficulties will appear and the generalization of the GWFA to a non-trivial background field will give the same results as the BTT. For instance, Ni *et al* [20] using the GWFA, obtained the same equation (60) for the sine–Gordon model. What is more beneficial, in [20] the authors found that equation (31) is approximately viable for a non-trivial background field. Of course, strictly speaking, the GWFA is difficult to embrace a non-uniform background field, because in this case equation (31) does not hold after all. This point may reflect the difference between the field-configuration-space formalism and the Fock-space formalism. We feel that the BTT can be regarded as the Fock-spatial version of the GWFA and so there is no equation (31)-like requirement for the BTT. Therefore, the BTT effectively avoids the difficulty of generalizing the GWFA to the non-uniform background case.

5. Conclusions

In this paper, we have compared the BTT with the GWFA by calculating the effective potential, one- and two-particle energies of the model (1), and generalizing the BTT to the non-trivial background case. In the constant background field case, both of the methods are equivalent to each other. For the space-dependent background, the BTT is more acceptable than the GWFA. It should be emphasized that the results in this paper are valid for those models in which the potentials have Fourier representations in the sense of tempered distributions. Therefore when such a model is investigated with the BTT or the GWFA, it is enough to directly employ the formulae in this paper, which will greatly simplify the calculations. As in the time-dependent generalization of the GWFA [31–34], the BTT can be directly generalized to a time-dependent formalism.

Finally, we would like to mention that in section 2, when $|\varphi\rangle$ takes the place of $|0\rangle$, the results do not change at all. This shows that if some Hermitian operator O is only linear or quadratic in the fields Φ's and their conjugate Π's (A_j in equation (5) is similar), the unitary transformation of the Gaussian wavefunctional $|\varphi\rangle$, i.e. $\mathrm{e}^{-\mathrm{i}sO}|\varphi\rangle$ (s is some parameter) does not produce non-Gaussian wavefunctional, which is consistent with that pointed out in [35]. Furthermore, take $g(p) = 0$ and $A_2 = 0$, and truncate the operator $\exp\{\int \mathrm{d}^D p\theta(p)a^\dagger(p,m)a^\dagger(-p,m)\}$, one can obtain the BCS-type vacuum state [36, 37].

Thus, from the aforementioned discussion, one can see that the BCS-type vacuum state may not lead to new results; however, generally this is so [36, 37].

Acknowledgments

This project was supported in part by the NSF of China. We thank the referees for their helpful comments.

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